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On zero varieties of holomorphic functions in Hardy spaces ☆

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Abstract

In contrast to the famous Henkin–Skoda theorem concerning the zero varieties of holomorphic functions in the Nevanlinna class on the open unit ball B_n in $\mathbb{C}^n$, $n \geqslant 2$, it is proved in this article that for any nonnegative, increasing, convex function $\varphi(t)$ defined on $\mathbb{R}$, there exists $g \in \mathcal{O}(B_n)$ satisfying $\int_S \varphi(N_g(\zeta,1))\,d\sigma(\zeta) < \infty$ such that there is no $f \in H^p(B_n)$, $0 < p < \infty$, with $\mathcal{Z}(f) = \mathcal{Z}(g)$. Here $N_g(\zeta,1)$ denotes the integrated zero counting function associated with the slice function g_ζ. This means that the zero sets of holomorphic functions belonging to the Hardy spaces $H^p(B_n)$, $0 < p < \infty$, unlike that of the holomorphic functions in the Nevanlinna class, cannot be characterized in the above manner.

Keywords: Hardy spaces; Nevanlinna class; Blaschke condition

1. Introduction

Let $B_n = \{z \in \mathbb{C}^n \mid |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $n \geqslant 1$, and let $S = S_n = bB_n$ denote the boundary of B_n. The space of holomorphic functions on B_n will be denoted by $\mathcal{O}(B_n)$. Then, we define the Hardy spaces $H^p(B_n)$, $0 < p < \infty$, and the Nevanlinna class $N(B_n)$ as follows:

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$$H^p(B_n) = \left\{ f \in \mathcal{O}(B_n) \;\middle|\; \sup_{0<r<1} \left(\int_S \left| f(r\zeta) \right|^p d\sigma(\zeta) \right)^{1/p} < \infty \right\}, \tag{1.1}$$

and

$$N(B_n) = \left\{ f \in \mathcal{O}(B_n) \;\middle|\; \sup_{0<r<1} \int_S \log^+ \left| f(r\zeta) \right| d\sigma(\zeta) < \infty \right\}. \tag{1.2}$$

We also let

$$H^\infty(B_n) = \left\{ f \in \mathcal{O}(B_n) \;\middle|\; \|f\|_\infty < \infty \right\} \tag{1.3}$$

be the space of bounded holomorphic functions on B_n. Here $d\sigma$ is the normalized Lebesgue measure on S with $\sigma(S) = 1$. For instance, when $n = 1$, $d\sigma = \frac{1}{2\pi} d\theta$. Then, $H^\infty(B_n) \subset H^p(B_n) \subset H^q(B_n) \subset N(B_n)$, if $0 < p < q < \infty$.

If $f \in \mathcal{O}(U)$, where $U = B_1 \subset \mathbb{C}$, denote by $n_f(t)$ the number of zeros of f in tU, $0 < t < 1$. We assume from now on that $f(0) = 1$. The integrated zero counting function $N_f(r)$ is defined by

$$N_f(r) = \int_0^r \frac{n_f(t)}{t} \, dt. \tag{1.4}$$

It is well known that $N_f(r)$ can be expressed through Jensen's formula as

$$N_f(r) = \int_{-\pi}^{\pi} \log \left| f\left(re^{i\theta}\right) \right| d\sigma(\theta) = \log \frac{r^k}{\prod_{j=1}^k |a_j|}, \tag{1.5}$$

where $a_1, \ldots, a_k$ are the zeros of f located in rU. Thus, for any $f \in N(U)$, following from

$$N_f(1) = \sup_{0<r<1} \int_{-\pi}^{\pi} \log \left| f\left(re^{i\theta}\right) \right| d\sigma(\theta) < \infty, \tag{1.6}$$

the zero set $\mathcal{Z}(f) = \{a_j\}_{j=1}^\infty$ of f satisfies the Blaschke condition $\sum_{j=1}^\infty (1 - |a_j|) < \infty$. Conversely, any point sequence $\{a_j\}_{j=1}^\infty$ in U satisfying the Blaschke condition is the zero set of a bounded holomorphic function, for instance, the Blaschke product $B(z)$ of $\{a_j\}_{j=1}^\infty$. This shows that, if $n = 1$, the zero sets of holomorphic functions in the Nevanlinna class (and hence Hardy spaces or space of bounded holomorphic functions) can be completely characterized by the Blaschke condition.

For $n \geqslant 2$, let $\zeta \in S$ be a boundary point. Denote by $f_\zeta(z) = f(\zeta z)$, $z \in \mathbb{C}$, the slice function associated with $f \in \mathcal{O}(B_n)$. Then define $n_f(\zeta, t)$ and $N_f(\zeta, t)$ associated with f_ζ in the above manner. Hence

$$N_f(\zeta, r) = \int_0^r \frac{n_f(\zeta, t)}{t} \, dt = \int_{-\pi}^{\pi} \log \left| f\left(re^{i\theta}\zeta\right) \right| d\sigma(\theta). \tag{1.7}$$

The counting function $N_f(\zeta, r)$ increases as r increases. Thus, if $\varphi(r)$ is a nonnegative, increasing, convex function on $\mathbb{R}$, one may apply Jensen's inequality to (1.7), and integrate it over S to get

$$\int_S \varphi\bigl(N_f(\zeta, 1)\bigr)\, d\sigma(\zeta) \leqslant \sup_{0<r<1} \int_S \varphi\bigl(\log\bigl|f(r\zeta)\bigr|\bigr)\, d\sigma(\zeta). \tag{1.8}$$

If $\varphi(t) = e^{pt}$, $0 < p < \infty$, (1.8) leads to a necessary condition of the counting function $N_f(\zeta, 1)$ for $f \in H^p(B_n)$, i.e.,

$$\int_S e^{pN_f(\zeta,1)}\, d\sigma(\zeta) \leqslant \sup_{0<r<1} \int_S \bigl|f(r\zeta)\bigr|^p\, d\sigma(\zeta) < \infty; \tag{1.9}$$

and if $\varphi(t) = t^+ = \max\{0, t\}$, (1.8) gives

$$\int_S N_f(\zeta, 1)\, d\sigma(\zeta) \leqslant \sup_{0<r<1} \int_S \log^+\bigl|f(r\zeta)\bigr|\, d\sigma(\zeta) < \infty \tag{1.10}$$

for $f \in N(B_n)$.

It is also well known that $f \in N(B_n)$ if and only if

$$\sup_{0<r<1} \int_S \bigl|\log\bigl|f(r\zeta)\bigr|\bigr|\, d\sigma(\zeta) < \infty. \tag{1.11}$$

For a proof of (1.11) the reader is referred to, e.g., Rudin [8]. Hence, for any $f \in N(B_n)$, we have the finiteness of the right-hand side of the following equation:

$$\int_S N_f(\zeta, 1)\, d\sigma(\zeta) = \sup_{0<r<1} \int_S \log\bigl|f(r\zeta)\bigr|\, d\sigma(\zeta) < \infty, \tag{1.12}$$

which, through the equality, carries full information of the zero set of f. This observation explains why the finiteness of the right-hand side of (1.12) can also serve as a sufficient condition for the zero set of $f \in N(B_n)$. That is, we have the following famous theorem due to G.H. Henkin and H. Skoda.

Theorem 1.1 (Henkin–Skoda). *If $f \in \mathcal{O}(B_n)$ satisfies*

$$\sup_{0<r<1} \int_S \log\bigl|f(r\zeta)\bigr|\, d\sigma(\zeta) < \infty, \tag{1.13}$$

then there exists $g \in N(B_n)$ such that $\mathcal{Z}(g) = \mathcal{Z}(f)$.

Here the notation $\mathcal{Z}(f)$ means the zero variety of f. The proof of Theorem 1.1 can be found in Henkin [7] and Skoda [11]. The Henkin–Skoda theorem for the Nevanlinna class was originally proved on any smooth bounded strongly pseudoconvex domains in $\mathbb{C}^n$, $n \geqslant 2$. Later, it was generalized to some other classes of domains, for instance, see Bruna, Charpentier and Dupain [2] for convex domains of finite type in $\mathbb{C}^n$; Chang, Nagel, and

Stein [4] for finite type domains in $\mathbb{C}^2$; and Shaw [10] for real ellipsoids in $\mathbb{C}^n$. See also Anderson [1] and Charpentier [5] for some related results.

When $\varphi(t) = e^{pt}$, $0 < p < \infty$, the right-hand side of (1.9) has now been adopted as the definition of the Hardy spaces $H^p(B_n)$. Obviously, lots of information concerning the zero set of $f \in H^p(B_n)$ were missing in the estimate (1.9) when Jensen's inequality was applied. It is the evidence that indicates that the definition of the Hardy spaces might not carry the full power description of the zero varieties of the holomorphic functions belonging to it. In order to carry the full information of the zero variety of $f \in H^p(B_n)$, one might expect that there is a nonnegative, increasing function $\varphi_{f,\zeta}(t)$ on $\mathbb{R}$ associated with f satisfying the following equality:

$$\begin{aligned}\int_S \varphi_{f,\zeta}\big(N_f(\zeta,1)\big)\,d\sigma(\zeta) &= \sup_{0<r<1}\int_S \exp\big(p\log\big|f(r\zeta)\big|\big)\,d\sigma(\zeta)\\ &= \sup_{0<r<1}\int_S \big|f(r\zeta)\big|^p\,d\sigma(\zeta) < \infty.\end{aligned}$$

Thus, also in view of the Henkin–Skoda's theorem on the Nevanlinna class, we shall make the following definition.

Definition 1.2. Let $X(B_n)$, e.g., $H^\infty(B_n)$, $H^p(B_n)$ or $N(B_n)$, be a subspace of $\mathcal{O}(B_n)$, and let $\varphi(t)$ be a nonnegative, increasing, convex function defined on $\mathbb{R}$. It is said that $\varphi(t)$ defines either a necessary or a sufficient condition for holomorphic functions in $X(B_n)$, if the following conditions are satisfied:

(I) necessary condition for $X(B_n)$:

$$\int_S \varphi\big(N_f(\zeta,1)\big)\,d\sigma(\zeta) < \infty \quad \text{for every } f \in X(B_n);$$

(II) sufficient condition for $X(B_n)$: for any $g \in \mathcal{O}(B_n)$ with

$$\int_S \varphi\big(N_g(\zeta,1)\big)\,d\sigma(\zeta) < \infty,$$

there is $f \in X(B_n)$ such that $\mathcal{Z}(f) = \mathcal{Z}(g)$.

If both conditions (I) and (II) are satisfied by $\varphi(t)$ for $X(B_n)$, then it is said that $\varphi(t)$ defines a necessary and sufficient conditions for the zero sets of holomorphic functions in $X(B_n)$.

Obviously, $\varphi(t) = e^{pt}$ defines a necessary condition for $H^p(B_n)$. Meanwhile, the Henkin–Skoda theorem states that, according to the Definition 1.2, the function $\varphi(t) = t^+ = \max\{0,t\}$ characterizes completely the zero sets of holomorphic functions in the Nevanlinna class.

The purpose of this article is to prove the following main result.

Theorem 1.3. *If $n \geqslant 2$, for a fixed $p > 0$, there is no nonnegative, increasing, convex function $\varphi(t)$ defined on $\mathbb{R}$ such that both* (I) *and* (II) *hold for $H^p(B_n)$ simultaneously. In other words, the zero sets of the holomorphic functions in the Hardy space $H^p(B_n)$ cannot be characterized in the above sense.*

As a matter of fact, what we are going to do is to show that the sufficient condition (II) fails for the Hardy spaces $H^p(B_n)$. Namely, we shall prove the following theorem.

Theorem 1.4. *Let $\varphi(t)$ be a nonnegative, increasing, convex function defined on $\mathbb{R}$. Then there exists $g \in \mathcal{O}(B_n)$ such that*

$$\int_S \varphi\bigl(N_g(\zeta, 1)\bigr)\, d\sigma(\zeta) < \infty,$$

and that there is no $f \in H^p(B_n)$, $0 < p < \infty$, satisfying $\mathcal{Z}(g) \subset \mathcal{Z}(f)$.

When $\varphi(t) = e^{pt}$, the result can be found in Rudin [8]. One should also compare the above results with those stated in Chang [3] and Varopoulos [12] using uniform Blaschke condition.

2. A special subspace $\mathcal{O}^p(U)$ of $H^p(B_2)$

Let D be a domain in $\mathbb{C}^n$, $n \geqslant 1$. For $0 < p < \infty$, define $\mathcal{O}^p(D) = \mathcal{O}(D) \cap L^p(D)$. Then we have

Theorem 2.1. *The following estimates hold*:

(1) *if $f \in H^p(B_n)$, $n \geqslant 1$, $0 < p < \infty$, then*

$$\bigl|f(z)\bigr| \leqslant 2^{n/p}\|f\|_{H^p}\bigl(1 - |z|\bigr)^{-n/p}, \quad \textit{for every } z \in B_n;$$

(2) *if $f \in \mathcal{O}^p(U)$, $0 < p < \infty$, then*

$$\bigl|f(z)\bigr| \leqslant \pi^{-1/p}\|f\|_{L^p}\bigl(1 - |z|\bigr)^{-2/p}, \quad \textit{for every } z \in U.$$

For any function $f \in \mathcal{O}(B_n)$, define the restriction $\mathcal{R}f$ of f on $\mathcal{O}(B_{n-1})$ by $\mathcal{R}f(z'') = f(z'', 0)$, where $z'' = (z_1, \ldots, z_{n-1})$. Conversely, if $g \in \mathcal{O}(B_{n-1})$, we define the extension $\mathcal{E}g$ of g on $\mathcal{O}(B_n)$ by $\mathcal{E}g(z'', z_n) = g(z'')$. The following theorem is well known, for instance, see Rudin [8].

Theorem 2.2. *If $0 < p < \infty$ and $n \geqslant 2$, then*

(1) *the extension $\mathcal{E}$ is a linear isometry from $\mathcal{O}^p(B_{n-1})$ into $H^p(B_n)$*;
(2) *the restriction $\mathcal{R}$ is a linear norm-decreasing map from $H^p(B_n)$ onto $\mathcal{O}^p(B_{n-1})$.*

Now, for any $\eta > 0$, define

$$D_\eta = \bigl\{z \in U \bigm| 1 - |z| < \eta\bigl(1 - |2z - 1|\bigr)\bigr\}. \tag{2.1}$$

Clearly, $D_{\eta_1} \subset D_{\eta_2}$ if $0 < \eta_1 < \eta_2$. In particular,

$$D_1 = \left\{z \in U \,\middle|\, \left|z - \frac{2}{3}\right| < \frac{1}{3}\right\}. \tag{2.2}$$

It is also easily seen that if $z \in D_\eta$, then

$$|2z - 1| < 1 - \frac{1 - |z|}{\eta} < 1.$$

This implies $2z - 1 \in U$ if $z \in D_\eta$. Denote by $e^{i\theta} D_\eta = \{e^{i\theta} z \mid z \in D_\eta\}$ for $\theta \in \mathbb{R}$. Then we prove the following theorem.

Theorem 2.3. *Let $\{a_j\}_{j=1}^\infty$ be a sequence of points approaching a boundary point $p_0 = e^{i\theta_0}$ within the region $e^{i\theta_0} D_\eta$ in the unit disc U for some $\eta > 0$. Assume $\sum_{j=1}^\infty (1 - |a_j|) = \infty$ and denote $E = \{(z_1, z') \in B_n \mid z_1 = a_j$ for some $j\}$ where $z' = (z_2, \ldots, z_n)$. If $f \in \mathcal{O}(B_n)$ satisfies the growth condition*

$$\bigl|f(z)\bigr| < \exp\left(\frac{c}{1 - |z|^2}\right)^\alpha, \quad z \in B_n,$$

for some $c < \infty$ and some $\alpha < \frac{1}{2}$, and if $E \subset \mathcal{Z}(f)$, then $f \equiv 0$.

Theorem 2.3 generalizes the results proved in Shapiro–Shields [9] and Rudin [8] where the theorem was proved when $\{a_j\}_{j=1}^\infty$ are lying on the same radius.

Proof. First, we may assume that $p_0 = (1, 0)$. Since $\{a_j\}$ approaches z_0 within D_η, the above calculation shows that $2a_j - 1 \in U$ for all $j \in \mathbb{N}$.

Now, following the notations and proof of Theorem 7.3.4 presented in Rudin [8], the function g is in the Nevanlinna class $N(U)$ and vanishes at $2a_j - 1$ for all j. The hypothesis on $\{a_j\}$ and the definition of the domain D_η now shows that $\sum_{j=1}^\infty (1 - |2a_j - 1|) = \infty$. Therefore, $g \equiv 0$, and hence $f \equiv 0$. This completes the proof of Theorem 2.3. □

Following from Theorems 2.1, 2.2, and 2.3, we immediately get

Theorem 2.4. *Let $\{a_j\}_{j=1}^\infty$ be a sequence of points approaching a boundary point $p_0 = e^{i\theta_0}$ within the region $e^{i\theta_0} D_\eta$ in the unit disc U of the complex plane for some $\eta > 0$. Then the set $E = \bigcup_{j=1}^\infty \{(a_j, z_2) \mid z_2 \in \mathbb{C}\} \cap B_2$ is the zero set of a nontrivial $f \in H^p(B_2)$, $0 < p < \infty$, obtained from an extension of some $g \in \mathcal{O}^p(U)$, i.e., $f = \mathcal{E} g$, if and only if $\sum_{j=1}^\infty (1 - |a_j|) < \infty$.*

Note that the order of contact of the region D_η with the line $x = 1$ is two at the point $(1, 0)$. Therefore, the sequence of points $\{a_j\}_{j=1}^\infty$ are allowed to approach the boundary point $p_0 = e^{i\theta_0}$ tangentially in Theorems 2.3 and 2.4. When the points $\{a_j\}_{j=1}^{\infty}$ stays inside a Stolz region, i.e., they approach the boundary point nontangentially, a proof of the theorem can be found (Theorem 4.14) in [6]. Since $\mathcal{O}^p(U)$ can be identified with a proper subspace of $H^p(B_2)$ by (1) of Theorem 2.2, it is of interest itself to see if the zero set of f in $\mathcal{O}^p(U)$ can be completely characterized or not. So far, some interesting results have been obtained in this direction. However, it is still far away from a complete understanding of the zero set of f in $\mathcal{O}^p(U)$. For a good reference of this aspect the reader is referred to [6].

3. Proofs of the main results

Proof of Theorem 1.4. First, we assume that $\varphi(t)$ is a C^1, nonnegative, increasing, convex function defined on $\mathbb{R}$. Thus, the function $\varphi(e^t)$ is a C^1, nonnegative, increasing, convex function defined on $\mathbb{R}$ with inverse function given by $(\varphi(e^t))^{-1} = \ln(\varphi^{-1})(t)$, where φ^{-1} is the inverse function of φ. Note that φ^{-1} is a C^1, concave, increasing function. Let k_0 be a positive integer such that

$$\begin{aligned} &(1) \quad \varphi^{-1}(\ln k_0) \geqslant e, \quad \text{and} \\ &(2) \quad \big(\varphi^{-1}\big)'(\ln t) \leqslant \varphi^{-1}(\ln t) \quad \text{for all } t \geqslant k_0. \end{aligned}$$

Then, we set for $k \geqslant k_0$,

$$x_k = 1 - \frac{h(k)}{k},$$

where

$$h(t) = \max\left\{\frac{1}{t}, t\frac{d}{dt}\big(\ln\big(\varphi^{-1}(\ln t)\big)\big)\right\} = \max\left\{\frac{1}{t}, \frac{(\varphi^{-1})'(\ln t)}{\varphi^{-1}(\ln t)}\right\}.$$

Thus, for $t \geqslant k_0$, $h(t)$ is a continuous, decreasing function satisfying $\lim_{t\to\infty} h(t) = 0$.

Following from the integral test, since

$$\int_{k_0}^{\infty} \frac{h(t)}{t}\,dt \geqslant \int_{k_0}^{\infty} \frac{(\varphi^{-1})'(\ln t)}{t\varphi^{-1}(\ln t)}\,dt = \ln\big(\varphi^{-1}(\ln t)\big)\Big|_{k_0}^{\infty} = \infty,$$

we get

$$\sum_{k=k_0}^{\infty} \frac{h(k)}{k} = \infty.$$

It follows that $\sum_{k=k_0}^{\infty}(1 - x_k) = \infty$.

Now, let g be a holomorphic function in $\mathcal{O}(B_n)$ such that $g(z) = g(z_1, 0)$ and that the zero set of g is given by $\mathcal{Z}(g) = \{(z_1, z') \in B_n \mid z_1 = x_k \text{ for some } k \geqslant k_0\}$. The existence of such g is obvious. Then, following from Theorems 2.1 and 2.3, there is no holomorphic function f in the Hardy spaces $H^p(B_n)$ such that f vanishes on $\mathcal{Z}(g)$.

Next, we calculate the counting function $n_g(\zeta, t)$ of g. First, we do it on the unit disc. That is, for each $0 < t < 1$, we want to know the number of x_k's such that $x_k < t$. Hence, we are seeking for the largest integer $k \geqslant k_0$ satisfying

$$\frac{k}{h(k)} < \frac{1}{1-t}, \tag{3.1}$$

and

$$\frac{k+1}{h(k+1)} \geqslant \frac{1}{1-t}. \tag{3.2}$$

It follows from the choice of $h(t)$ that we have

$$(k+1)^2 \geqslant \frac{k+1}{h(k+1)} \geqslant \frac{1}{1-t}. \tag{3.3}$$

Since $h(t)$ is a decreasing function, we get from (3.1) and (3.3) that

$$k < \frac{1}{1-t} h(k) \leqslant \frac{1}{1-t} h\left(\sqrt{\frac{1}{1-t}} - 1\right). \tag{3.4}$$

It follows that for $\zeta = (\zeta_1, \dots, \zeta_n) \in S$ with $|\zeta_1| \geqslant |\zeta_1^0|$ and $t > T_0$, we have the estimate for the counting function

$$n_g(\zeta, t) \leqslant \frac{1}{1-t|\zeta_1|} h\left(\sqrt{\frac{1}{1-t|\zeta_1|}} - 1\right). \tag{3.5}$$

The constants $|\zeta_1^0|$ and T_0 will be chosen sufficiently close to 1 with the following properties:

(1) $T_0|\zeta_1^0| \geqslant \frac{4}{5}$,

(2) $\varphi^{-1}\left(\ln\left(\frac{1}{\sqrt{1-T_0|\zeta_1^0|}} - 1\right)\right) \geqslant e$, and

(3) $\frac{5}{2} \ln \varphi^{-1}(\ln x) + 10 \leqslant \varphi^{-1}(\ln x)$, for $x \geqslant \frac{1}{\sqrt{1-|\zeta_1^0|}} - 1$.

We shall also assume that k_0 is chosen so that $x_{k_0} > T_0$. Thus, for $\zeta \in S$ with $|\zeta_1| \geqslant |\zeta_1^0|$, we get

$$\begin{aligned} N_g(\zeta, 1) &= \int_0^1 \frac{n_g(\zeta, t)}{t}\, dt = \int_{T_0}^1 \frac{n_g(\zeta, t)}{t}\, dt \\ &\leqslant \int_{T_0}^1 \frac{1}{t(1-t|\zeta_1|)} h\left(\sqrt{\frac{1}{1-t|\zeta_1|}} - 1\right) dt \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{T_0}\int_{T_0}^{1}\frac{1}{(1-t|\zeta_1|)}\left\{\frac{(\varphi^{-1})'\Big(\ln\Big(\sqrt{\frac{1}{1-t|\zeta_1|}}-1\Big)\Big)}{\varphi^{-1}\Big(\ln\Big(\sqrt{\frac{1}{1-t|\zeta_1|}}-1\Big)\Big)}+2\sqrt{1-t|\zeta_1|}\right\}dt\\
&\leqslant \frac{2}{T_0|\zeta_1|}\int_{T_0}^{1}\frac{|\zeta_1|}{2(1-t|\zeta_1|-(1-t|\zeta_1|)^{3/2})}\frac{(\varphi^{-1})'\Big(\ln\Big(\sqrt{\frac{1}{1-t|\zeta_1|}}-1\Big)\Big)}{\varphi^{-1}\Big(\ln\Big(\sqrt{\frac{1}{1-t|\zeta_1|}}-1\Big)\Big)}\,dt\\
&\quad+\frac{2}{T_0}\int_{T_0}^{1}\frac{1}{\sqrt{1-t|\zeta_1|}}\,dt\\
&\leqslant \frac{2}{T_0|\zeta_1|}\ln\varphi^{-1}\Big(\ln\Big(\frac{1}{\sqrt{1-t|\zeta_1|}}-1\Big)\Big)\Big|_{T_0}^{1}+\frac{4}{T_0}(1-T_0)^{1/2}\\
&\leqslant \frac{5}{2}\ln\varphi^{-1}\Big(\ln\Big(\frac{1}{\sqrt{1-|\zeta_1|}}-1\Big)\Big)+10\leqslant\varphi^{-1}\Big(\ln\Big(\frac{1}{\sqrt{1-|\zeta_1|}}-1\Big)\Big).
\end{aligned}
$$

Therefore, from the estimate of $N_g(\zeta,1)$, we immediately get

$$
\begin{aligned}
\int\limits_{\substack{S\\|\zeta_1^0|\leqslant|\zeta_1|}}\varphi\big(N_g(\zeta,1)\big)\,d\sigma(\zeta)&\leqslant\int\limits_{\substack{S\\|\zeta_1^0|\leqslant|\zeta_1|}}\varphi\Big(\varphi^{-1}\Big(\ln\Big(\frac{1}{\sqrt{1-|\zeta_1|}}-1\Big)\Big)\Big)\,d\sigma(\zeta)\\
&=\int\limits_{\substack{S\\|\zeta_1^0|\leqslant|\zeta_1|}}\ln\Big(\frac{1}{\sqrt{1-|\zeta_1|}}-1\Big)\,d\sigma(\zeta)<\infty.
\end{aligned}
$$

This completes the proof of Theorem 1.4 for the case when $\varphi\in C^1$.

For the general case, it is easily seen that there is a C^1 nonnegative, increasing, convex function φ_1 such that $\varphi(t)\leqslant\varphi_1(t)$ for all t. For instance, φ_1 can be obtained by convolving φ with $\chi(t)\in C_0^\infty(-1,1)$ satisfying $\chi\geqslant 0$, $\chi(t)=\chi(|t|)$, and $\int_{-1}^{1}\chi(t)\,dt=1$. Then, constructing g with respect to φ_1 as before, we immediately get

$$
\int\limits_S\varphi\big(N_g(\zeta,1)\big)\,d\sigma(\zeta)\leqslant\int\limits_S\varphi_1\big(N_g(\zeta,1)\big)\,d\sigma(\zeta)<\infty.
$$

The proof of Theorem 1.4, and hence of Theorem 1.3, is thus completed. □

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